Unsteady Potential and Viscous Flows between Eccentric Cylinders

W.-G. Sim* and Y.C. Cho**

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This paper presents a newly developed spectral collocation method for the study of the unsteady annular flow between two eccentric cylinders. In order to predict the stability of a system in a confined flow, the formulae and results of added mass and fluid damping are provided in the present paper when a cylinder undergoes oscillatory motion in the plane of symmetry and normal to the plane of symmetry in an eccentric annulus. The potential flow theory has been developed to obtain the added mass for incompressible, inviscid and irrotational fluid. For the viscous fluid, the added mass and the viscous damping are presented. This method is validated by comparison with the available analytical solutions obtained for the unsteady potential flow in the eccentric annular space. Excellent agreement was found between the solutions obtained with the present spectral method and the available analytical solutions. In the present study, the viscous effect on the added mass can be evaluated, comparing the results obtained by potential flow theory with those obtained by the viscous flow theory, and viscous damping is investigated.

Key Words: Flow-Induced Vibration, Added Mass, Viscous Damping, Hydrodynamic Force

1. Introduction

Cylindrical structures subjected to annular flow are widely used in many engineering constructions; *e. g.*, control rods in guide tubes of PWR-reactors, feed water spargers in BWR-type reactors, fuel-cluster stringers in AGR-type reactors, tubes in the baffle regions of some kinds of heat exchangers and certain types of valves and pistons. For sufficiently high flow velocities, the cylinders in such arrangements have often developed self-excited oscillations, sometimes severe and occasionally destructive. For this reason, increasingly more effort has recently been devoted to research in this area.

* Auxiliary Systems Design Dept., Korea Atomic Energy Research Institute, P.O. Box 7, Daeduk-Danji, Taejeon, Korea, 305-606

** Department of Mechanical Engineering, Inha University, Inchon, Korea

The dynamics and stability of a cylinder in confined flow represents a coupled fluid-structure interaction problem (Païdoussis and Ostoja-Starzewski, 1981; Païdoussis, et al., 1990). Hence, it is essential to formulate the hydrodynamic forces associated with the motion of the cylinder. In a linear analysis, the unsteady, motion-related, fluid-dynamic forces may be conveniently separated into inertia, damping and stiffness components. Therefore, it is a logical first step to develop analytical tools which may be used to predict the inertial added mass and damping forces in flow or just in quiescent fluid. As is well-known, added mass, in phase with the acceleration of the moving cylinder, and damping, in phase with its velocity, are dependent on fluid parameters and system geometry. Studies of added mass can be traced to Stokes(1843) and a brief survey was presented by Muga and Wilson(1970).

A considerable amount of work has been done on the dynamics of a cylinder immersed in stationary confined viscous or inviscid fluid. Fritz(1972) developed a method for calculating the inertial forces, in which an appropriate ideal flow solution was proposed, and then generalized forces were obtained via Lagrange's equations of motion. The fluid-dynamic forces acting on oscillating rods in stationary confined fluid have been studied by potential flow theory (Au-Yang, 1976; Chung and Chen, 1977) and by viscous theory (Chen, et al., 1976; Yeh and Chen, 1977; Mateescu, et al., 1989) based on the linearized Navier-Stokes equations of motion.

For narrow annular configurations, where the viscous damping is specially important in stationary confined fluid, three-dimensional effects on the hydrodynamic forces, considering the end effect due to finite length annular region where both ends of the annulus are open, have been studied by Mulcahy(1980) using simplified Navier-Stokes equations. The theory was formulated for various viscous penetration depths (to be defined later) in the narrow annular space. As the ratio of length to radius of the inner cylinder is decreased, this three-dimensional effect becomes significant.

Because of the special interest in the dynamics of systems involving eccentricities, sudden expansions, contractions or diffuser sections (Mulcahy, 1980; Hobson and Jedwab, 1990), and because even in the case of smooth concentric configurations it would be of interest to provide more accurate solutions for viscous flows than has heretofore been possible, recourse to numerical solution techniques had to be taken (Mateescu, et al., 1991). A comprehensive research effort in this direction has been initiatied, aiming to eventually obtain solutions for the dynamics of annular configurations involving generally variable annular spaces and concentric or eccentric cylinders oscillating in laminar or turbulent viscous flows.

This paper presents the first results of this research and is concerned with the numerical solutions for the steady and unsteady flows between eccentric cylinders. Eccentricities may actually appear in real engineering systems due to several technical causes, such as manufacturing and mounting errors, or deformations; the fluid-elastic interaction may also represent an important cause for eccentricities (eccentricities may appear as a result of flow-induced divergence-type instabilities or vibrations of the flexible or flexibly-mounted centre-bodies in such systems). The presence of an eccentricity in annular configurations considerably adds to the complexity of the problem and, for this reason, very few accurate analytical solutions were obtained, and this only for simplified cases and geometries involving eccentric cylinders.

The spectral method have first been applied to the unsteady potential flow and then to the unsteady viscous flow, generated by periodic translational motion of a cylinder in an eccentric annulus. In the present analysis, the problem is formulated based on the following assumptions: (a) the flow is two dimensional with no axial flow velocity and (b) the amplitude of the oscillatory motion of the cylinder is small. Comparing the results obtained by potential flow theory with those obtained by the viscous flow theory, the viscous effect on the added mass can be evaluated, and viscous damping is investigated. The results obtained by the potential flow theory will also be used to validate the present spectral method against the available analytical solutions of Chung and Chen(1977) for eccentric configurations and of Fritz(1972) for concentric configurations. To have meaningful comparison with the available solutions, the same considerations are used to solve this unsteady problem with the spectral collocation method.

2. General Considerations of the Numerical Method of Solution

Since, eventually, the problem of self-excited motions of flexible bodies in fluid flow in such geometries will have to be tackled, in which the motion of the cylinder in time and space is not known a priori, it is very important that the numerical method utilized be as computationally efficient and frugal (from both the memory requirement and time points of view) as possible. At the same time, the numerical method used for the fluid-dynamic problem should be consistent with modal analysis for the structural dynamics, which would facilitate the fluid-elastic stability analysis (a longer term goal) via a Galerkin-type technique (Païdoussis, et al., 1990). For these reasons, a spectral collocation method was adopted for the fluid-dynamic problem: the unsteady potential and viscous flows between two eccentric

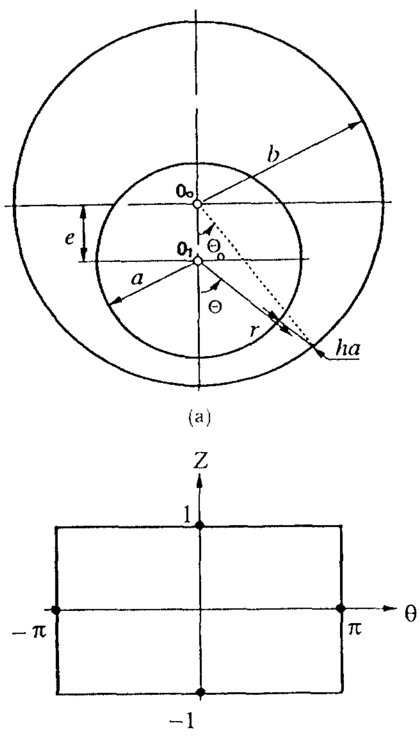

Fig. 1 Geometry of the annular space between two eccentric cylinder: (a) physical plane (r, Θ) and (b) computational domain (Z, θ) obtained by the coordinate transformations(1)

cylinders.

In such two-dimensional flow problems, any fluid-dynamic property, f, at any location in the annular space between the eccentric cylinders depends on the radial and circumferential (angular) coordinates r and θ, shown in Fig. 1, and eventually on the time, t, when the flow is unsteady.

A convenient coordinate transformation is first used to transform the annular space (r, Θ) between the eccentric cylinders into a rectangular computational domain (Z, θ), where the nondimensional coordinate Z is defined as:

$$Z=1-2\frac{r-a}{a\,h(\theta)}, \tag{1}$$

in which

$$a\,h(\theta)=R_0(\theta)-a=\sqrt{b^2-e^2\sin^2\theta}-e\cos\theta-a,\ \theta=\Theta. \tag{2}$$

In the above expressions a and b are the ratio of the inner and outer cylinders, and e denotes the eccentricity ($\check{e}$ represents the relative eccentricity with respect to the inner cylinder radius a)

In the present spectral method, the following type of expansion is considered for any fluid-dynamic property f in the two-dimensional annular space

$$f(Z,\ \theta,\ t)=\sum_j\sum_k\sum_l f_{jkl}T_j(Z)\,F_k(\theta)\exp(i\omega_l t), \tag{3}$$

where $T_j(Z)$ and $F_k(\theta)$ represent here Chebyshev polynomials and Fourier series functions, respectively. The choice of Fourier series for the interpolation functions in the circumferential direction, $F_k(\theta)$, stems from the obvious periodic character of the flow field with respect to θ, and allows a direct representation of the symmetry or antisymmetry of the flow variables with respect to the plane $\theta=0$ by using, accordingly, cosine or sine functions of $k\theta$. No such periodic character is obvious in the radial direction, hence the choice of the Chebyshev polynominals, $T_j(Z)$, for the corresponding interpolation function. The third expansion using complex exponential functions of time, which is introduced in Eq. (3) for the sake of generality, is obviously needed only in the unsteady flow case; as a result of this last expansion, the governing equations can be decomposed into N_t sets of equations, each set corresponding to a certain frequency ω_l, which can be solved serially.

The coefficients f_{jkl} of the flow variable expansions in the form (3), which are a priori unknown, have then to be determined from the governing equations of the steady or unsteady flow and the associated boundary conditions.

The present approach is based on collocation method which imposes that the governing equations, as well as the boundary conditions, are rigorously satisfied at a certain number of distinct locations within the computational domain, say $N_Z \times N_\theta$ locations; implicitly, it is assumed that at intermediate positions between the collocation points the governing equations are satisfied within a desired level of accuracy. The number of collocation points in the radial and circumferential directions, N_Z and N_θ, should be carefully selected to achieve the desired level of accuracy and, at the same time, good computing efficiency.

3. The Unsteady Potential Flow between Oscillating Eccentric Cylinders

When the viscosity of fluid, ν, is very small or the circular frequency of the motion of cylinder, ω, is very large, the penetration depth defined by $\delta_p = \sqrt{2\nu/\omega}$ is very small. In this case, the fluid flow can be assumed to be irrotatioal and inviscid. As mentioned before, the present spectral method will be validated against the available solutions (Chung and Chen, 1977; Fritz, 1972) which is based on an incompressible potential (inviscid) flow formulation.

3.1 Formulation of the basic equation

In this unsteady potential flow, the continuity equation may be expressed in terms of the unsteady velocity potential $\phi(r, \theta, t)$ and reduces to the Laplace equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0. \tag{4}$$

Assuming that the moving cylinder undergoes a harmonic oscillatory translation characterized by the radian frequency $\omega = 2\pi f$, the unsteady velocity potential can be expanded in the form

$$\phi(Z, \theta, t) = a^2 e^{i\omega t} \sum_{j=0}^{m} \sum_{k=0}^{n} \Phi_{jk} T_j(Z) f_k(\theta), \tag{5}$$

where the nondimensional coordinate Z is defined by coordinate transformation (1), and where $T_j(Z)$ represent the Chebyshev polynomials and $F_k(\theta)$ are Fourier series functions to be defined further.

With this expansion of the potential, the velocity potential Eq. (4) becomes

$$\sum_{j=0}^{m} \sum_{k=0}^{n} \Phi_{jk} \Big[A T_j''(Z) F_k(\theta) + B T_j'(Z) F_k(\theta) + C T_j'(Z) F_k'(\theta) + D T_j(Z) F_k''(\theta) \Big] = 0, \tag{6}$$

where A, B, C and D are functions of Z and $h(\theta)$ defined by the following equations;

$$\begin{aligned} A &= 1 + D[(1-Z) h'(\theta)/h(\theta)]^2, \\ B &= \bar{B}\sqrt{D} + D(1-Z)\{h''(\theta)/h(\theta) - 2 [h'(\theta)/h(\theta)]^2\}, \\ C &= 2D(1-Z) h'(\theta)/h(\theta), \\ D &= \{h(\theta)/[2 + (1-Z) h(\theta)]\}^2. \end{aligned} \tag{7}$$

To complete the problem formulation, we have to add to the above governing equation the boundary conditions on the moving and fixed cylinders, which are expressed in terms of the radial and circumferential velocity components, v and ω,

$$v = \frac{\partial \phi}{\partial r} = \frac{B_0}{a}\frac{\partial \phi}{\partial Z} = a B_0 e^{i\omega t} \sum_{j=0}^{m} \sum_{k=0}^{n} \Phi_{jk} T_j'(Z) F_k(\theta), \tag{8}$$

$$w = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{A_0}{a}\Big[\frac{\partial \phi}{\partial \theta} + (1-Z)\frac{h'(\theta)}{h(\theta)}\frac{\partial \phi}{\partial Z}\Big] = a A_0 e^{i\omega t} \sum_{j=0}^{m} \sum_{k=0}^{n} \Phi_{jk} \Big[T_j(Z) F_k'(\theta) + (1-Z)\frac{h'(\theta)}{h(\theta)} T_j'(Z) F_k(\theta) \Big], \tag{9}$$

where

$$\begin{aligned} A_0 &= 2/[2 + (1-Z) h(\theta)], \\ B_0 &= -2/h(\theta). \end{aligned} \tag{10}$$

(a) Oscillatory motions in the plane of symmetry $\theta=0$. Consider first the case when the inner cylinder is fixed and the axis of the outer cylinder executes an oscillatory translation in the plane $\theta=0$ containing the axes of the two cylinders, which will be referred to as the plane of symmetry. With $f_0(t)$ denoting the oscillatory displacement of the outer cylinder axis,

$$f_0(t)=a\ \hat{f}\ e^{i\omega t}, \tag{11}$$

the boundary conditions on the fixed and moving cylinders can be expressed (considering the small amplitude oscillation assumption) as

$$[v(r,\ \theta)]_{r=a}=0, \tag{12}$$

$$[v(r,\ \theta)\cos(\theta-\Theta)-w(r,\ \theta)\sin(\theta-\Theta_0)]_{r=b}=\frac{df_0}{dt}\cos\Theta_0, \tag{13}$$

where

$$\cos\Theta_0=[1+h(\theta)]\frac{a}{b}\cos\theta+\frac{e}{b}. \tag{14}$$

The above boundary conditions can be expressed in terms of the Chebyshev polynomials and Fourier series functions by using the Eqs. (8) and (9), in the form

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}T_j'(1)F_k(\theta)=0, \tag{15}$$

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}[C_0T_j'(-1)F_k(\theta)+D_0T_j(-1)F_k'(\theta)]=i\omega\ \hat{f}\ \cos\Theta_0, \tag{16}$$

where

$$C_0=B_0\cos(\theta-\Theta_0)+2D_0h'(\theta)/h(\theta),$$

$$D_0=-\{1/[1+h(\theta)]\}\sin(\theta-\Theta_0). \tag{17}$$

If the outer cylinder is fixed and the inner cylinder axis executes an oscillatory translation in the symmetry plane $\theta=0$ defined by the displacement

$$f_1(t)=a\ \hat{f}\ e^{i\omega t}, \tag{18}$$

the boundary conditions on the moving and fixed cylinder, respectively, can be expressed in the form

$$B_0\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}T_j'(1)F_k(\theta)=i\omega\ \hat{f}\ \cos\theta, \tag{19}$$

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}[C_0T_j'(-1)F_k(\theta)+D_0T_j(-1)F_k'(\theta)]=0. \tag{20}$$

It is obvious that in both cases when the oscillatory translation of the moving cylinder axis takes place in the plane of symmetry, the unsteady flow in the eccentric annular space is symmetric with respect to this plane $\theta=0$, and hence the Fourier functions $F_k(\theta)$ used in the expansion (5) of the velocity potential have to be defined as

$$F_k(\theta)=\cos k\theta. \tag{21}$$

The collocation method can be applied now, as described in Section 2, to the Eqs. (6) and (15), (16), or (19), (20), which will reduce to an algebraic system of equations leading to the solutions of the coefficients Φ_{jk} of the velocity potential expansion.

(b) Oscillatory motions normal to the plane of symmetry($\theta=0$). When the inner cylinder is fixed and the outer cylinder axis executes an oscillatory motion of translation perpendicular to the symmetry plane, $\theta=0$, defined by the displacement

$$g_0(t)=a\ \hat{g}\ e^{i\omega t}, \tag{22}$$

the boundary condition on the fixed inner cylinder in given by Eq. (15), while that on the moving outer cylinder can be expressed as

$$\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}[C_0T_j'(-1)F_k(\theta)+D_0T_j(-1)F_k'(\theta)]=i\omega\ \hat{g}\ \sin\Theta, \tag{23}$$

where C_0 and D_0 have the same expressions given in Eq. (17).

Similarly, when the outer cylinder is fixed and the inner cylinder axis has an oscillatory displacement normal to the symmetry plane $\theta=0$,

$$g_1(t)=a\ \hat{g}\ e^{i\omega t}, \tag{24}$$

the boundary condition on the fixed outer cylinder is given by Eq. (20), while that on the moving inner cylinder can be expressed as

$$B_0\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}T_j'(1)F_k(\theta)=i\omega\ \hat{g}\ \cos\theta. \tag{25}$$

In both these cases, when the oscillatory translation of the moving cylinder axis is normal to the plane $\theta=0$, the unsteady flow in the eccentric annular space cna be considered (when the oscillation amplitude is small) antisymmetric with respect to this plane $\theta=0$; hence, in these case the Fourier functions $F_k(\theta)$ used in the velocity potential expansion (5) have to be defined as sine functions of $k\theta$,

$$F_k(\theta)=\sin k\theta. \tag{26}$$

The above boundary conditions together with the governing Eq. (6) will also be reduced in these cases to an algebraic system of equations, leading to the solutions for the coefficients Φ_{jk} of the velocity potential expansion.

3.2 Unsteady pressure distribution and resultant pressure force

With the cofficients Φ_{jk} determined as indicated above, the entire flow field in the eccentric annular space is also completely determined. The unsteady pressure may be calculated now in the annular space from the Bernoulli-Lagrange equation

$$p-p_0=-\rho\left[\frac{\partial\phi}{\partial t}+\frac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{2}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2\right], \tag{27}$$

where p_0 denotes the stagnation pressure in the fluid (existing in the absence of any oscillatory motion of cylinders). Assuming small amplitude oscillations (in order to have a meaningful comparison with Chung and Chen's(1976) results), the unsteady pressure can be expressed in simple harmonic oscillation form,

$$p-p_0=\rho\omega^2a^2\bar{p}(Z,\ \theta)\,\varepsilon e^{i\omega t}, \tag{28}$$

where $\bar{p}(Z,\ \theta)$ is a nondimensional reduced pressure defined as

$$\bar{p}(Z,\ \theta)=\frac{1}{i\omega\varepsilon}\sum_{j=0}^{m}\sum_{k=0}^{n}\Phi_{jk}T_j(Z)F_k(\theta), \tag{29}$$

and where $\varepsilon=\hat{f}$ if the moving cylinder axis is oscillating in the plane of symmetry, or $\varepsilon=\hat{g}$ if it has oscillatory translations normal to the plane of symmetry.

The resultant unsteady force acting on the moving cylinder can be obtained by integrating the unsteady pressure along the circumference of the cylinder. For example, the resultant unsteady force per unit length acting on the inner cylinder

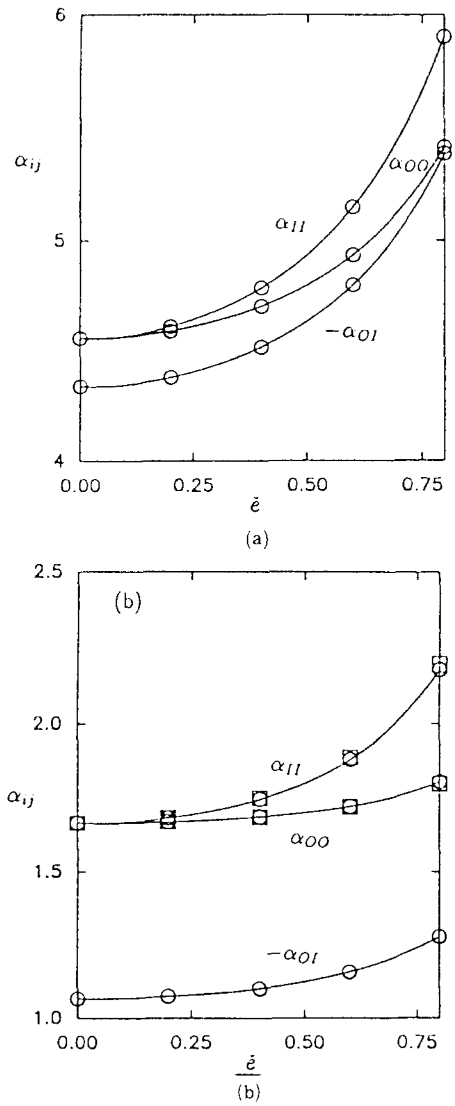

Fig. 2 The added mass coefficients, α_{II}, α_{OO} and α_{OI}, for oscillations in the plane of symmetry, as functions on the relative eccentricity $\check{e}=e/(b-a)$ for the cases: (a) $b/a=1.25$ and (b) $b/a=2$. Comparison between the present solution(–○–) and Chung and Chen's solution(□)

when its axis oscillates in the plane of symmetry is calculated as

$$F_I(t) = -\int_0^{2\pi} a\ p(r,\ \theta)\ \cos\theta\ d\theta. \quad (30)$$

Let α_{ij} denote the nondimensional coefficient of the resultant unsteady force acting on the cylinder i when the cylinder j axis oscillates in the plane of symmetry, with the convention that i and j are I for the inner cylinder and O for the outer one, which is defined in the form

$$\alpha_{ij} = 2F_i(t) \Big/ \left[\pi\rho\omega^2(r_i^2 + r_j^2)\, a\ \hat{f}\ e^{i\omega t}\right], \quad (31)$$

where $r_I = a$ and $r_O = b$. Similarly, let β_{ij} denote the nondimensional unsteady force coefficient of the cylinder i when the cylinder j axis has an oscillatory translation normal to the symmetry plane,

$$\beta_{ij} = 2F_i(t) \Big/ \left[\pi\rho\omega^2(r_i^2 + r_j^2)\, a\ \hat{g}\ e^{i\omega t}\right]. \quad (32)$$

The nondimensional unsteady force coefficients α_{ij} (also known as the added mass coefficients in the analysis of the dylinder cynamics) calculated with the present spectral collocation method as a function of the relative eccentricity $\check{e}$ are compared in Fig. 2 with the results obtained by Chung and Chen(1976).

The agreement between the present solution and Chung and Chen's results is very good (for the sake of this comparison, the same assumptions as those made by Chung and Chen were also used in the application of the present spectral collocation method).

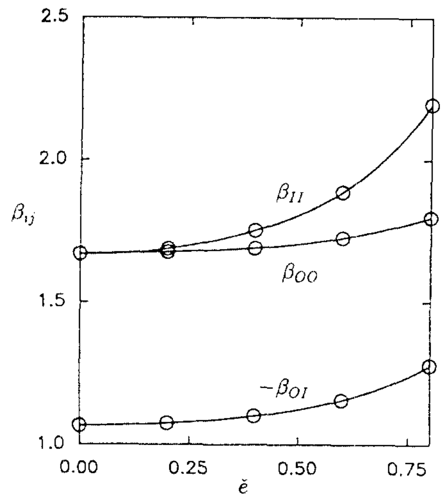

Fig. 3 The added mass coefficients, β_{II}, β_{OO} and β_{OI}, for oscillations normal to the symmetry plane, as functions of the relative eccentricity $\check{e} = e/(b-a)$ for $b/a = 2$

The variation with the relative eccentricity $\check{e}$ of the nondimensional unsteady force coeffcients β_{ij} in the case of oscillatory translations normal to the symmetry plane is shown for $b/a = 1.25$ in Fig. 3; one can notice that this variation is almost identical with that of the coefficients α_{ij} in Fig. 2(a), although the corresponding resultant un-

Table Variation of the calculated mass cofficients α_{II} and α_{IO} with the number of collocation points, m, and their relative difference with respect to Fritz's(1972) analytical results

b/a	m	α_{II}	$-\alpha_{IO}$	Comparison with Fritz's analytical results	
				$1-\alpha_{II}/\alpha_{IIFr}$	$1-\alpha_{IO}/\alpha_{IOFr}$
1.25	3	4.3117	4.1457	5.35%	4.39%
	5	4.5549	4.3355	0.01%	0.01%
	7	4.5556	4.3360	0.0001%	0.0001%
2	5	1.6409	1.0564	1.54%	0.96%
	7	1.6654	1.0662	0.07%	0.04%
	9	1.6666	1.0666	0.0001%	0.0001%

steady forces act in directions perpendicular to each other.

The influence of the number of collocation points on the accuracy of this spectral method is shown in Table by comparison with the analytical results obtained for concentric configurations ($e=0$) and small amplitude oscillations by Fritz (1972). It was found that the rate of convergence is faster in the case of the narrower annulus than the wider one. Also the difference between the numerical results and the analytical results appears to decrease faster with an increasing number of collocation points, m As a result, a slight increase in number of terms m. taken in calculation is found to be needed for larger b/a in order to obtain the same accuracy. The agreement between the numerical and analytical results is very good.

4. Unsteady Viscous Flow between Oscillating Eccentric Cylinders

Although for many engineering applications, the viscosity is small and the fluid may be considered inviscid as a first approximation, near the surface of cylinder there exists a thin layer of rotational flow, associatied with the penetration depth. This flow region, where the viscous effect is significant, is of great concern to the dynamic response of the system for annular configurations. In particular, when the annular gap is small, the viscous effect becomes pronounced. Thus, it is of interest to obtain the hydrodynamic force including viscous effect.

The hydrodynamic forces acting on the inner cylinder, due to the oscillatory motion of the inner cylinder, will be obtained through line integration of stresses and pressures around the circumference of the cylinder. In general for this problem, the resultant hydrodynamic forces have simple harmonic forms, based on the assumption of small amplitude oscillations, and are decomposed into two parts, one in phase with the acceleration and the other with the velocity of the motion.

4.1 Formulation of the basic equations

We consider the inner cylinder of the system, surrounded by viscous but incompressible fluid, and undergoing periodic translational motion in an eccentric annulus. The motion of the inner cylinder is assumed to be simple harmonic with circular frequency, ω and its amplitude small. For this kind of two dimensional problem without steady axial flow, it is possible to eliminate the convective terms and the axial component terms from the governing equations for unsteady fluid flow. The linearized Navier-Stokes equations and continuity equation in cylindrical coordinates can be reduced to

$$\frac{\partial w^*}{\partial t}+\frac{1}{\rho}\frac{1}{r}\frac{\partial p^*}{\partial \Theta}=\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w^*}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 w^*}{\partial \Theta^2}-\frac{w^*}{r^2}+\frac{2}{r^2}\frac{\partial v^*}{\partial \Theta}\right],$$

$$\frac{\partial v^*}{\partial t}+\frac{1}{\rho}\frac{\partial p^*}{\partial \Theta}=\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v^*}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v^*}{\partial r^2}-\frac{v^*}{r^2}-\frac{2}{r^2}\frac{\partial w^*}{\partial \Theta}\right], \tag{33}$$

$$\frac{\partial w^*}{\partial \Theta}+\frac{\partial}{\partial r}(rv^*)=0, \tag{34}$$

where v^* and w^* denote the unsteady flow velocities in the radial and circumferential directions, respectively.

Based on the no-slip condition at the interface between fluid and cylinder, the boundary conditions on the fixed ($r=b$) and moving ($r=a$) cylinders can be expressed, in cases of oscillatory motion (a) in the plane of symmetry and (b) normal to the plane of symmetry, as

$$v^*(b,\ \Theta)=w^*(b,\ \Theta)=0,$$

$$v^*(a,\ \Theta)=e_v\cos\Theta=\frac{e_I}{\partial t}\cos\Theta,$$

$$w^*(a,\ \Theta)=-e_v\sin\Theta=-\frac{e_I}{\partial t}\sin\Theta,$$

in case(a),

$$v^*(a,\ \Theta)=g_v\sin\Theta=\frac{g_I}{\partial t}\sin\Theta,$$

$$w^*(a,\ \Theta)=g_v\cos\Theta=\frac{g_I}{\partial t}\cos\Theta,$$

in case(b), (35)

where e_v and g_v represent the lateral velocity of the vibrating inner cylinder in cases (a) and (b), respectively, and e_I and g_I denote the corresponding displacement of the moving cylinder.

In order to generalize the present problem, it is convenient to define the following nondimensional parameters

$$\hat{v}=\frac{v^*}{\iota a\omega\varepsilon e^{\iota\omega t}},\qquad \hat{w}=\frac{w^*}{\iota aw\varepsilon e^{\iota wt}},$$
$$\hat{p}=\frac{p^*}{\rho a^2\omega^2\varepsilon e^{\iota wt}},\qquad h=\frac{H}{a},$$
$$\hat{e}=\frac{e_v}{\iota a\omega e^{\iota wt}},\qquad \hat{g}=\frac{g_v}{\iota a\omega\varepsilon e^{\iota wt}},$$
$$Re_s=\frac{\omega a^2}{\nu}=2\left(\frac{a}{\delta_p}\right)^2, \tag{36}$$

where $\hat{e}$ and $\hat{g}$ denote the nondimensional amplitueds of the displacement of the inner cylinder oscillating and ε stands for them ($\varepsilon=\hat{f}$ when the inner cylinder has oscillatory motion in the plane of symmetry, or $\varepsilon=\hat{g}$ when it has oscillatory motion normal to the plane of symmetry), and δ_p may be interpreted as the depth of penetration of viscous wave regarded as a unsteady fluid layer.

Considering the coordinate transformation with the above nondimensional parameter, it is not difficult to reformulate the governing equations and continuity Eqs. (33) and (34) in the computational domain (Z, θ) as a nondimensional form

$$\iota\frac{Re_s}{4}h^2\hat{w}-\iota\frac{Re_s}{2}h\sqrt{D}L(\hat{p})=\Big[A\frac{\partial^2\hat{w}}{\partial Z^2}+B\frac{\partial\hat{w}}{\partial Z}+C\frac{\partial^2\hat{w}}{\partial Z\partial\theta}+D\frac{\partial^2\hat{w}}{\partial\theta^2}-D(\hat{w}-2L(\hat{v}))\Big],$$
$$\iota\frac{Re_s}{4}h^2\hat{v}-\iota\frac{Re_s}{2}h\frac{\partial\hat{p}}{\partial Z}=\Big[A\frac{\partial^2\hat{v}}{\partial Z^2}+B\frac{\partial\hat{v}}{\partial Z}+C\frac{\partial^2\hat{v}}{\partial Z\partial\theta}+D\frac{\partial^2\hat{v}}{\partial\theta^2}-D(\hat{v}-2L(\hat{w}))\Big], \tag{37}$$

$$\frac{\partial\hat{v}}{\partial Z}-\sqrt{D}\,\hat{v}-\sqrt{D}L(\hat{w})=0, \tag{38}$$

where the operator $L(f)$ is

$$L(f)=\Big[\frac{\partial}{\partial\theta}+(1-Z)\frac{h'(\theta)}{h(\theta)}\frac{\partial}{\partial Z}\Big]f.$$

In this spectral method, the nondimensional fluid parameters can be expressed in terms of Chebyshev polynomials and Fourier expansions, as shown in the potential theory. By inspection of the boundary conditions and considering the properties of symmetry and antisymmetry of fluid parameters with respect to the plane of symmetry $\Theta=0$, the fluid parameters can be expressed only in even terms ($\cos k\theta$) or only in odd terms ($\sin k\theta$) of Fourier expansions, according to the direction of the oscillatory motion of cylinder; in case (a) or in case (b), as mentioned before.

(a) Oscillatory motions in the plane of symmetry, $\Theta=0$. Using the spectral expansion for the oscillatory motion of the inner cylinder in the plane of symmetry, the following types of expansions can be considered for the fluid-dynamic properties in two dimensional annular space

$$\hat{w}=\sum_{j=0}^{m}\sum_{k=0}^{n}W_{jk}T_j(Z)\,s(k\theta),$$
$$\hat{v}=\sum_{j=0}^{m}\sum_{k=0}^{n}V_{jk}T_j(Z)\,c(k\theta),$$
$$\hat{p}=\sum_{j=0}^{m-2}\sum_{k=0}^{n}P_{jk}T_j(Z)\,c(k\theta), \tag{39}$$

where $c(k\theta)$ and $s(k\theta)$ stand for the even terms $\cos k\theta$ and odd terms $\sin k\theta$ of the Fourier expansions, respectively, and the unknown coefficients W_{jk}, V_{jk} and P_{jk} are in complex forms due to the viscosity.

Taking account of the expansion forms shown in the above equations, the governing equations and continuity equation can be expanded as

$$\sum_{j=0}^{m}\sum_{k=0}^{n}W_{jk}\Big[AT_j''(Z)s(k\theta)+BT_j'(Z)s(k\theta)+CT_j'(Z)s'(k\theta)+DT_j(Z)s''(k\theta)-DT_j(Z)s(k\theta)-\iota\frac{Re_s}{4}h^2T_j(Z)s(k\theta)\Big]$$
$$+2V_{jk}\Big[T_j(Z)c'(k\theta)+(1-Z)\frac{h'(\theta)}{h(\theta)}T_j'(Z)c(k\theta)\Big]+\iota P_{jk}\frac{Re_s}{2}h(\theta)\sqrt{D}\Big[T_j(Z)c'(k\theta)+(1-Z)\frac{h'(\theta)}{h(\theta)}T_j'(Z)c(k\theta)\Big]=0,$$

$$\sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk}\Big[A T_j''(Z)\, c(k\theta) + BT_j'(Z)\, c(k\theta) + CT_j'(Z)\, c'(k\theta) + DT_j(Z)\, c''(k\theta) - DT_j(Z)\, c(k\theta) - \iota\frac{Re_s}{4} h^2 T_j(Z)\, c(k\theta)\Big] - 2W_{jk}\Big[T_j(Z)\, s'(k\theta) + (1-Z)\frac{h'(\theta)}{h(\theta)} T_j'(Z)\, s(k\theta)\Big] + \iota P_{jk}\frac{Re_s}{2} h(\theta)\, T_j'(Z) c(k\theta) = 0, \tag{40}$$

$$\sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk}\Big[T_j'(Z)\, c(k\theta) - \sqrt{D}\, T_j(Z)\, c(k\theta)\Big] - W_{jk}\Big[T_j(Z)\, s'(k\theta) + (1-Z)\frac{h'(\theta)}{h(\theta)} T_j'(Z)\, s(k\theta)\Big] = 0, \tag{41}$$

subject to the boundary conditions

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk} T_j(1)\, c(k\theta) = \cos\theta,\\
&\sum_{j=0}^{m}\sum_{k=0}^{n} W_{jk} T_j(1)\, s(k\theta) = -\sin\theta,\\
&\sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk} T_j(-1)\, c(k\theta) = \cos 0,\\
&\sum_{j=0}^{m}\sum_{k=0}^{n} W_{jk} T_j(-1)\, c(k\theta) = \cos 0,
\end{aligned} \tag{42}$$

here ( )′ and ( )″ denote the first and second order differentiations, respectively, with respect to the concerned parameter; for example, $T' = \partial T/\partial Z$ and $c'' = \partial^2 c/\partial^2\theta$. In the present analysis, the unknown coefficients can be determined by the collocation method, whereby the governing equations and continuity equation are satisfied at a certain number of distinct locations within the computational domain, say $(3m-1)\times(n+1)$. As a result, the discretized set of equations can be obtained from the governing equation, in addition to the boundary conditions $2\times(n+1)$. Thus, the solutions of the algebraic system of $(3m+1)\times(n+1)$ equations can be obtained completely in the computational domain, which are convertible back to the physical domain.

(b) Oscillatory motions normal to the plane of symmetry $\Theta=0$. In the spectral expansion when the inner cylinder has oscillatory motion normal to the plane of symmetry, while the outer cylinder is fixed, the following type of expansions can be considered for the fluid-dynamic properties in the two-dimensional annular space by inspection of the boundary conditions and the properties of symmetry and antisymmetry of fluid parameters:

$$\begin{aligned}
\hat{w} &= \sum_{j=0}^{m}\sum_{k=0}^{n} W_{jk} T_j(Z)\, c(k\theta),\\
\hat{v} &= \sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk} T_j(Z)\, s(k\theta),\\
\hat{p} &= \sum_{j=0}^{m-2}\sum_{k=0}^{n} P_{jk} T_j(Z)\, s(k\theta),
\end{aligned} \tag{43}$$

in terms of unknown coefficients, W_{jk}, V_{jk} and P_{jk} which are separated into real and imaginary components.

Considering the above equations in expansion form, the first of the two Navier-Stokes equations can be written as

$$\sum_{j=0}^{m}\sum_{k=0}^{n} W_{jk}\Big[A T_j''(Z)\, c(k\theta) + BT_j'(Z)\, c(k\theta) + CT_j'(Z)\, c'(k\theta) + DT_j(Z)\, c''(k\theta) - DT_j(Z)\, c(k\theta) - \iota\frac{Re_s}{4} h^2 T_j(Z)\, c(k\theta)\Big] + 2V_{jk}\Big[T_j(Z)\, s'(k\theta) + (1-Z)\frac{h'(\theta)}{h(\theta)} T_j'(Z)\, s(k\theta)\Big] + \iota P_{jk}\frac{Re_s}{2} h(\theta)\sqrt{D}\Big[T_j(Z)\, s(k\theta) + (1-Z)\frac{h'(\theta)}{h(\theta)} T_j'(Z)\, s(k\theta)\Big] = 0, \tag{44}$$

subject to the boundary conditions

$$\begin{aligned}
&\sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk} T_j(1)\, s(k\theta) = \sin\theta,\\
&\sum_{j=0}^{m}\sum_{k=0}^{n} W_{jk} T_j(1)\, c(k\theta) = \sin\theta,\\
&\sum_{j=0}^{m}\sum_{k=0}^{n} V_{jk} T_j(-1)\, s(k\theta) = 0,\\
&\sum_{j=0}^{m}\sum_{k=0}^{n} W_{jk} T_j(-1)\, s(k\theta) = 0,
\end{aligned} \tag{45}$$

while the other Navier-Stokes equations and continuity equation are not given here for brevity.

Similarly to case (a), imposing the governing equations and continuity equation on a certain number of collocation points and considering the boundary conditions, the discretized algebraic equations can be obtained in closed form, to determine the unknown coefficients, V_{jk}, W_{jk} and P_{jk}. Considering the obtained coefficients, which are the complex, the fluid dynamic parameters in the physical domain can be evaluated completely by the coordinate transformations.

4.2 Shear stress and resultant viscous forces

The hydrodynamic forces, which can be separated into self-added mass and viscous damping terms, acting on the inner cylinder can be calculated by line integration of the stress components including pressure. For the present analysis in cylindrical coordinates, the stress component can be rewritten as

$$\tau_{rr} = -p^* + 2\mu\frac{\partial v^*}{\partial r},$$
$$\tau_{r\Theta} = \mu\left\{\frac{\partial w^*}{\partial r} + \frac{w^*}{r} + \frac{1}{r}\frac{\partial v^*}{\partial \Theta}\right\}, \tag{46}$$

where the unsteady pressure and unsteady flow velocities are determined in terms of Chebyshev polynomials and Fourier expansions through the spectral collocation method.

The resultant forces, acting on the inner cylinder per unit length, in the direction of oscillatory motion can be calculated by circumferential integration of the stress components on the wall as

$$F_I = \int_0^{2\pi} a(\tau_{rr}|_{r=a}\cos\Theta - \tau_{r\Theta}|_{r=a}\sin\Theta)\,d\Theta,$$
$$G_I = \int_0^{2\pi} a(\tau_{rr}|_{r=a}\sin\Theta + \tau_{r\Theta}|_{r=a}\cos\Theta)\,d\Theta, \tag{47}$$

where F_I and G_I stands for the cases (a) and (b), respectively, and the stress components on the surface of the inner cylinder can be expanded in complex form. Thus the hydrodynamic forces can be separated into real and imaginary components. Substituting Eq. (46) into Eq. (47), these forces can be expressed in the form

$$F_I = -\rho\pi a^2 C_M \frac{\partial^2 e_I}{\partial t^2} - C_v \frac{\partial e_I}{\partial t} = \rho\pi a^2\omega^2 a\bar{e}e^{\iota\omega t}[\Re(\hat{F}) + \iota\Im(\hat{F})],$$
$$G_I = -\rho\pi a^2 C_M \frac{\partial^2 g_I}{\partial t^2} - C_v \frac{\partial g_I}{\partial t} = \rho\pi a^2\omega^2 a\hat{g}e^{\iota\omega t}[\Re(\hat{G}) + \iota\Im(\hat{G})], \tag{48}$$

where C_M and C_v represent the added mass and viscous damping coefficients, respectively, and e_I and g_I denote the displacements of the moving cylinder in cases (a) and (b), respectively. From Eq. (48), by definition, the added mass and damping coefficients can be written as

$$C_M = \Re(\hat{F}),\ C_v = -\rho\pi a^2\omega\Im(\hat{F}), \quad \text{in case (a)},$$

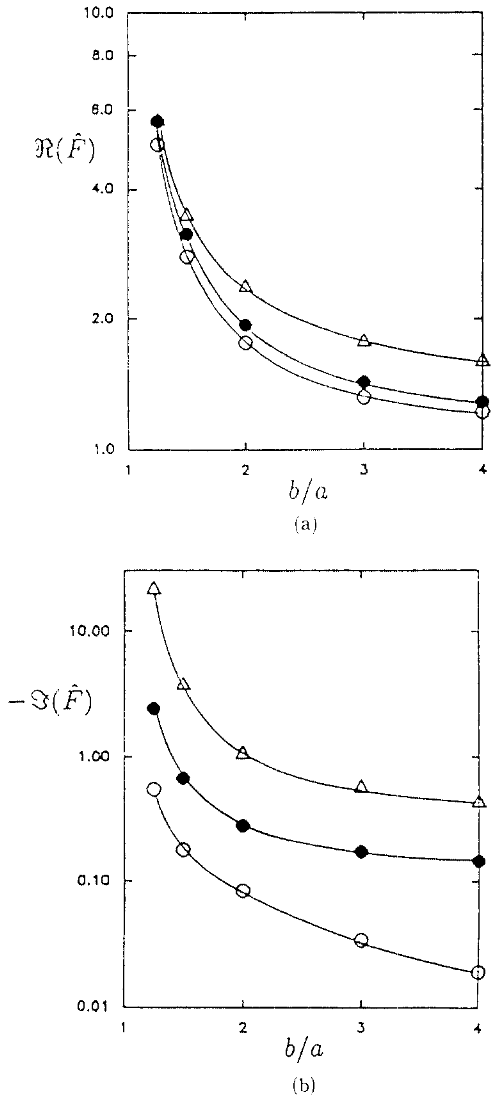

Fig. 4 The (a) real and (b) imaginary components of the nondimensional fluid-dynamic forces versus the radius ratio, b/a, for the selected oscillatory Reynolds number: △, $Re_s = 50$; ●, $Re_s = 500$; ○, $Re_s = 5000$

$$C_M = \Re(\hat{G}),\ C_v = -\rho\pi a^2\omega\Im(\hat{G}), \quad \text{in case (b)}. \quad (49)$$

4.3 Numerical results for viscous fluid-dynamic forces

To illustrate the influence of viscosity of the fluid on added mass and the viscous damping for the problem of harmonic oscillatory motion of the inner cylinder in an eccentric annulus, the calculations have been conducted while varying the oscillatory Reynolds number, Re_s, the ratio of radii, b/a, and relative eccentricity, $e/(b-a)$.

In these calculations, the collocation points (m

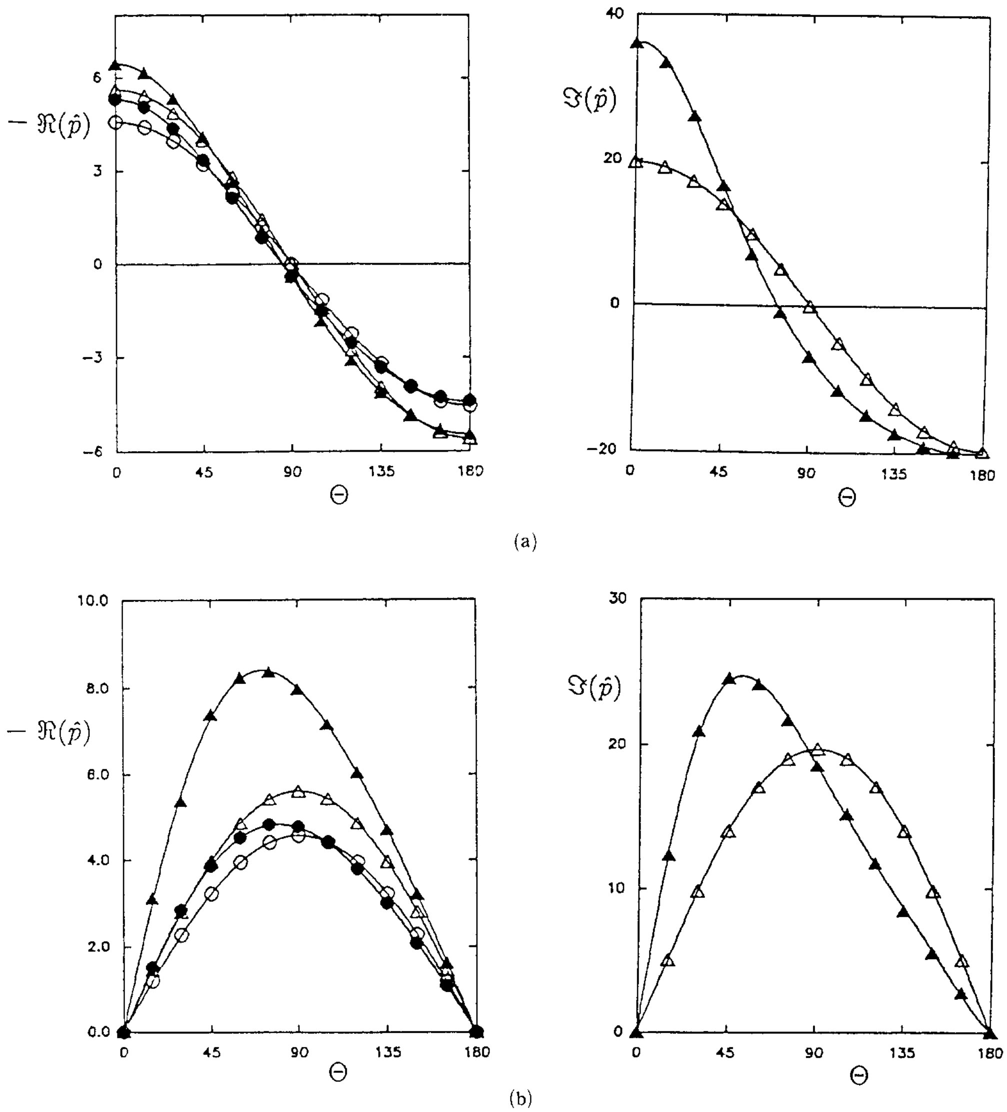

Fig. 5 Influence of eccentricity on the nondimensional pressure, $\hat{p}$ obtained by the present potential (○, ●) and viscous (△, ▲ ; $Re_s = 50$) theories in the case of $b/a = 1.25$ for oscillations; (a) in the plane of symmetry and (b) normal to the symmetry plane. Open symbols, $\check{e} = e/(b-a) = 0$; filled symbols, $\check{e} = 0.4$

-1) along the radial direction are clustered near the wall to obtain good accuracy and computing effciency, when the penetration depth is relatively small vis-à-vis the annular space, $\delta_p/(b-a)<0.1$. In other cases, the calculations have been conducted with equally distributed collocation points along the radial direction. Along the circumferential direction, equally distributed collocation points ($n+1$) are selected, but with $F_k(\theta)\neq 0$ to avoid the pseudo-singularity problem.

When Re_s is 50, 500 and 5000, and b/a is varied from 1.25 to 4, the added mass and viscous damping coefficients for concentric configurations are shown in Fig. 4. It is found that the

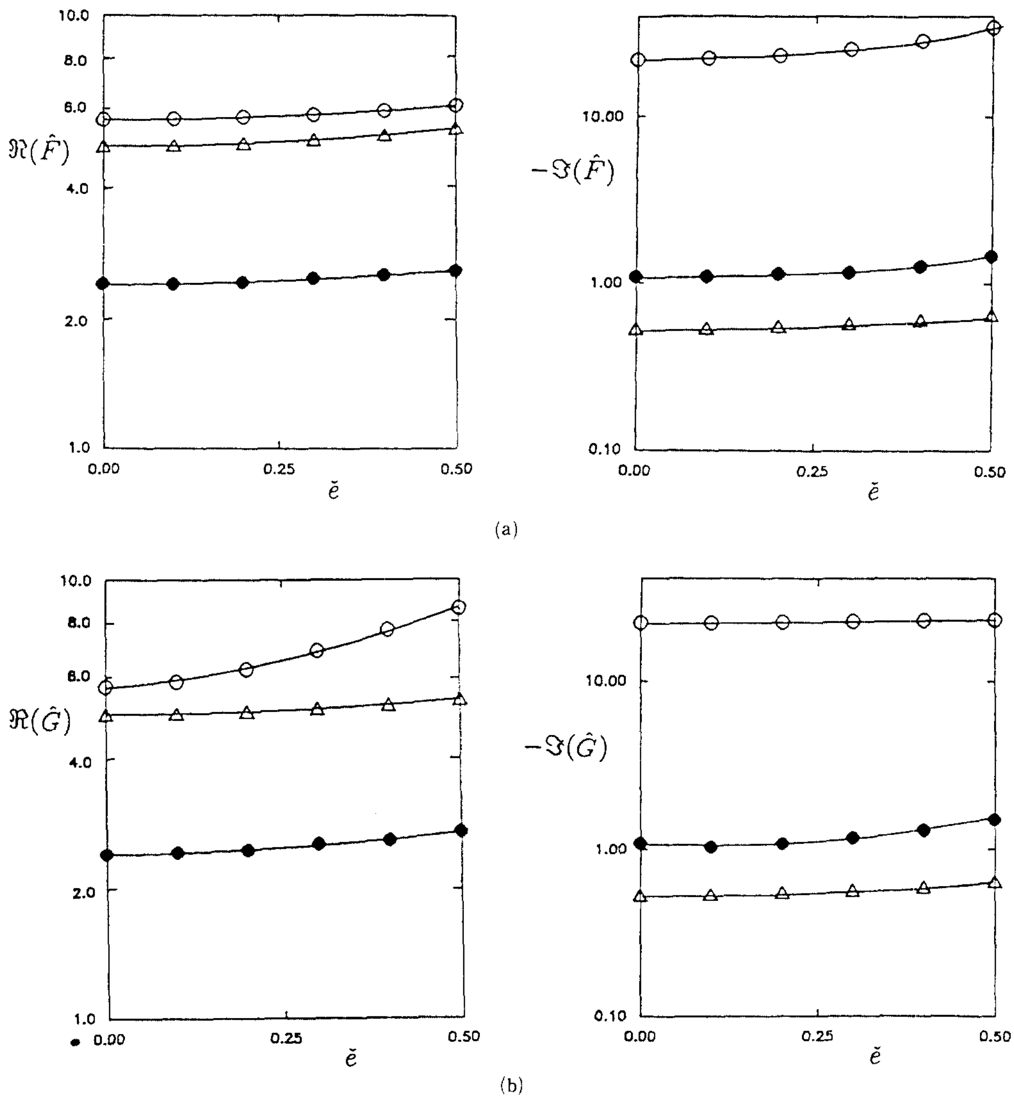

Fig. 6 Influence of the relative eccentricity $\check{e}=e/(b-a)$ on the nondimensional fluid-dynamic forces considering full viscous effects for oscillations; (a) in the plane symmetry and (b) normal to the symmetry plane. $-\circ-$, $Re_s=50$ and $b/a=1.25$; $-\bullet-$, $Re_s=50$ and $b/a=2$; $-\triangle-$, $Re_s=5000$ and $b/a=1.25$

coefficients are strongly dependent on the oscillatory Reynolds number; as it increases, these coefficients decrease. Physically, for fixed values of the ratio of radii, b/a, and the viscosity of fluid, ν these coeffcients decrease with increasing the frequency of oscillatory motion, ω. The two coefficients exponentially increase with decreasing b/a for the fixed oscillatory Reynolds number. Particularly for narrow annular flow, it is necessary to take into account the viscous damping, even if the oscillatory Reynolds number is high, corresponding to the case of low viscosity fluid or high circular frequency. With increasing the value of oscillatory Reynolds number, the added mass coefficients is less influenced by viscosity of fluid, and not too different from the result obtained by the potential flow theory.

The influence of the relative eccentricity on the nondimensional pressure in complex form is illustrated in Fig. 5(a) for $b/a=1.25$ and $Re_s=50$ in the case of oscillatory motion in the plane of symmetry and in Fig. 5(b) in the case of oscillatory motion normal to the plane of symmetry. The real part of it, which is related to the added mass, is compared with the result (open circles for concentric configurations and filled circles for eccentric ones $e/(b-a)=0.4$) for potential flow. The character of the variation of $\Re(\bar{p})$ and $\Im(\bar{p})$ with the eccentricity is similar to that for potential flow.

The added mass and viscous damping coefficients are shown in Fig. 6(a) for the oscillatory motion in the plane of symmetry and Fig. 6(b) for the motion normal to the plane of symmetry. The relative eccentricity, $e/(b-a)$, effect on the coefficients is investigated with the selected oscillatory Reynold number ($Re_s=50$, 5000) and the ratio of radii ($b/a=1.25$, 2). The numerical results have been calculated with $m<6$ and $n<6$ in case of $Re_s=50$ and with $m<10$ and $n<4$ in case of $Re_s=5000$, in order to minimize the round-off error which may increase with the size of the matrix obtained from the algebraic equations. In general, it is necessary to increase the terms in the Fourier expansion with increasing eccentricity, and of Chebyshev polynomials with increasing of the annular space. As the eccentricity increases, the magnitude of these coefficients increases and, due to the viscosity, the added mass coefficients increase as the oscillatory Reynolds number decreases.

5. Conclusions

A newly developed spectral collocation method is presented in this paper for the study of unsteady potential and viscous flows between fixed and oscillatory cylinders eccentrically positioned.

The spectral method is applied in this paper to the unsteady potential and viscous flows, generated by the harmonic translational motion of cylinder in an annulus based on small amplitude motion. The numerical results are presented to evaluate the general characteristics of the added masses for both flows and the viscous damping for viscous flow in terms of the radius ratio b/a with the eccentricity $e/(b-a)$. For viscous flow, the oscillatory Reynolds number Re_s is an important parameter, as shown in Eq. (37).

To assess the validity of the results for potential flow, the present results are compared with the analytical results given by Chung and Chen (1977) for eccentric configurations and by Fritz (1972) for concentric ones. Excellent agreement was found in both cases between the solution obtained with the present spectral method and the available analytical solutions. One can conclude that the present spectral collocation method has been validated by these comparisons, and we can safely proceed further to use it for solving more complicated unsteady flow problems, which remain unsolved at present. The numerical results for both potential and viscous flows are compared. The difference between the two sets of results can be explained by the viscous effects caused by the shear stress and the unsteady pressure drop in circumferential direction.

Considering the results obtained by potential and viscous flow theories for translational motion of the inner cylinder in an annulus, the following remarks should be made: (a) the present collocation method has been validated by comparison with the analytical ones. Therefore, this method can be adapted for use in more complicated unsteady flow problems, which remain unsolved at present; (b) the linear theory presented in this analysis is based on the assumption of small oscillatory amplitudes (as a result, the added mass and viscous damping coefficients are independent of the amplitude); (c) the added mass and viscous damping coefficients are dependent of the oscillatory Reynolds number, and these coefficients are influenced by the relative eccentricity; with decreasing oscillatory Reynolds number and increasing the eccentricty, these coefficients increase; (d) for the high oscillatory Reynolds number, the added mass coefficients can be estimated approximately by potential flow theory, but the viscous damping coefficients, even for high oscillatory Reynolds number, should be considered in the hydrodynamic forces for narrow annuli; (e) for narrow configurations, the added mass is insensitive to variations of the oscillatory Reynolds number; however, the damping is sensitive to it.

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